

Cross-Intersecting Erdős-Ko-Rado Sets in Finite Classical Polar Spaces

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ABSTRACT. A cross-intersecting Erdős-Ko-Rado set of generators of a finite classical polar space is a pair (Y, Z) of sets of generators such that all $y \in Y$ and $z \in Z$ intersect in at least a point. We provide upper bounds on $|Y| \cdot |Z|$ and classify the cross-intersecting Erdős-Ko-Rado sets of maximum size with respect to $|Y| \cdot |Z|$ for all polar spaces except Hermitian polar spaces in odd projective dimension.

1. Introduction

Erdős-Ko-Rado sets (*EKR sets*) were introduced by Erdős, Ko, and Rado [6] as a set Y of k -element subsets of $\{1, \dots, n\}$ such that the elements of Y pairwise intersect non-trivially. In particular, Erdős, Ko, and Rado partially classified all such Y of maximum size.

THEOREM 1.1 (Theorem of Erdős, Ko, and Rado). *Let be $n \geq 2k$. Let Y be an EKR set of k -element subsets of $\{1, \dots, n\}$. Then*

$$|Y| \leq \binom{n-1}{k-1}$$

with equality for $n > 2k$ if and only if Y is set of all k -elemental sets containing a fixed element.

Stronger versions of this theorem were later proven by several authors including the famous work by Wilson [19], Ahlswede and Khachatrian [1].

This theorem for EKR sets was generalized to many structures, including subspaces of projective spaces [12, 7, 3] and generators (maximal totally isotropic, respectively, singular subspaces) of polar spaces [16, 3, 4]. In polar spaces the problem is partially open, since the maximum size of EKR sets of generators of $H(2d-1, q^2)$, $d > 3$ odd, is still unknown. To the knowledge of the author the best known upper bound is given in [13].

There exists the following modification of the original problem which generated a lot of interest: a *cross-intersecting EKR set* is a pair (Y, Z) of sets of subsets with k elements of $\{1, \dots, n\}$ such that all $y \in Y$ and $z \in Z$ intersect non-trivially. If one wants to generalize the theorem of Erdős, Ko, and Rado to this structure, then the following question arises: how do we measure the size of (Y, Z) ? There are at least two natural choices. Either one goes for an upper bound for $|Y| + |Z|$ or one tries to find the upper bound for $|Y| \cdot |Z|$. In the set case the first project was pursued in [9], while the second one was completed in [15]. Results for vectors spaces are due to Tokushige [17]. Again this problem can be generalized to polar spaces, where

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an *cross-intersecting EKR set of generators* is a pair (Y, Z) of sets of generators such that all $y \in Y$ and $z \in Z$ intersect in at least a point. In this setting this paper is only concerned with an upper bound for $|Y| \cdot |Z|$ and a classification of all cross-intersecting EKR sets reaching this bound.

One additional motivation for this problem is the following: as mentioned before the problem of EKR sets of maximum size in $H(2d - 1, q^2)$ is still open for $d > 3$ odd. Let P be a point of $H(2d - 1, q^2)$ and let X be an EKR set of $H(2d - 1, q^2)$. Furthermore, let Y be the set of generators of X on P and Z the set of generators of X not on P . Now in the quotient geometry of P isomorphic to $H(2d - 3, q^2)$ the projection of the generators of Y and Z onto the quotient geometry is a cross-intersecting EKR set. So both problems are related.

One last thing to point out is that this work does not provide tight upper bounds for cross-intersecting EKR sets in $H(2d - 1, q^2)$ for all $d > 1$. The problem is very similar to the open problem of the maximum size of EKR sets in $H(9, q^2)$. Therefore, it could be reasonable to first solve the problem of the maximum size of cross-intersecting EKR sets in $H(7, q^2)$ and then generalize the technique to EKR sets in $H(9, q^2)$.

2. Projective Spaces & Polar Spaces

We refer to [10] for details on *projective spaces*. A projective space $\text{PG}(n - 1, q)$ of *projective dimension* $n - 1$ (respectively *vector space dimension* n) over the field with q elements has exactly

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \prod_{i=1}^k \frac{q^{n-i+1} - 1}{q^i - 1}$$

subspaces of (vector space) dimension k . We denote the number of points in $\text{PG}(n - 1, q)$ by

$$[n]_q := \begin{bmatrix} n \\ 1 \end{bmatrix}_q.$$

So we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^k \frac{[n - i + 1]_q}{[i]_q}.$$

We shall write $\begin{bmatrix} n \\ k \end{bmatrix}$ instead of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ whenever the choice for q is clear. We will often use the following analog of the recursive definition of binomial coefficients.

$$(2.1) \quad \begin{bmatrix} n + 1 \\ k + 1 \end{bmatrix} = \begin{bmatrix} n \\ k + 1 \end{bmatrix} + q^{(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}$$

REMARK 2.2. All the used eigenvalue formulas are more convenient if we use vector space dimensions and not projective dimensions. Consequently, the word *dimension* will always refer to the vector space dimension of a subspace.

A *polar space* is a incidence geometry with subspaces of dimension from 0 to d defined by a non-degenerate sesquilinear form or a non-degenerate quadratic form. The *finite classical polar spaces* are $Q^+(2d - 1, q)$, $Q(2d, q)$, $Q^-(2d + 1, q)$, $W(2d - 1, q)$, where q is a prime power, $H(2d - 1, q)$, and $H(2d, q)$, where q is the square of a prime power. We refer to [11] for details. Denote totally isotropic, respectively, singular subspaces of (vector space) dimension d as *generators*. Each subspace of (vector space) dimension $d - 1$ of a polar space is incident with exactly $q^e + 1$ generators, where $e = 0$ for $Q^+(2d - 1, q)$, $e = 1/2$ for $H(2d - 1, q)$, $e = 1$ for

$Q(2d, q)$ and $W(2d-1, q)$, $e = 3/2$ for $H(2d, q)$, and $e = 2$ for $Q^-(2d+1, q)$. It is well known that a polar space possesses exactly

$$\prod_{i=0}^{d-1} (q^{i+e} + 1)$$

generators and $(q^{d+e-1} + 1)[d]$ points (i.e. 1-dimensional totally isotropic, respectively, singular subspaces).

3. The Association Scheme of a Polar Space

We need some basic properties of an association scheme of generators on a dual polar space of rank d and type e . A complete introduction to association schemes can be found in [2, Ch. 2].

DEFINITION 3.1. *Let X be a finite set. A d -class association scheme is a pair (X, \mathcal{R}) , where $\mathcal{R} = \{R_0, \dots, R_d\}$ is a set of symmetric binary relations on X with the following properties:*

- (1) $\{R_0, \dots, R_d\}$ is a partition of $X \times X$.
- (2) R_0 is the identity relation.
- (3) There are numbers p_{ij}^k such that for $x, y \in X$ with xR_ky there are exactly p_{ij}^k elements z with xR_iz and zR_jy .

The number $n_i := p_{ii}^0$ is called the i -valency of R_i . The total number of elements of X is

$$n := |X| = \sum_{i=0}^d n_i.$$

The relations R_i are described by their adjacency matrices $A_i \in \mathbb{C}^{n,n}$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xR_iy \\ 0 & \text{otherwise.} \end{cases}$$

There exist (e.g. in [2, p. 45]) idempotent Hermitian matrices $E_j \in \mathbb{C}^{n,n}$ (hence they are positive semidefinite) with the properties

$$\begin{aligned} \sum_{j=0}^d E_j &= I, & E_0 &= n^{-1}J, \\ A_j &= \sum_{i=0}^d P_{ij}E_i, & E_j &= \frac{1}{n} \sum_{i=0}^d Q_{ij}A_i, \end{aligned}$$

where $P = (P_{ij}) \in \mathbb{C}^{d+1, d+1}$ and $Q = (Q_{ij}) \in \mathbb{C}^{d+1, d+1}$ are the so-called eigenmatrices of the association scheme.

The generators of a polar space define an association scheme if we say that two generators a and b are in relation R_i if and only if $\text{codim}(a \cap b) = i$. Hence a cross-intersecting EKR set (Y, Z) is a set of vertices such that there are no edges between Y and Z in the (distance-regular) graph associated with A_d . This scheme is *cometric*, so there exists a natural ordering of its E_j 's and its eigenspaces W_j [2, Sec. 2.7, Sec. 9.4]. The matrix P can be found in the literature (for example in [18, Theorem 4.3.6]). In particular, the eigenvalues of A_d are

$$(3.2) \quad (-1)^r q^{\binom{d-r}{2} + \binom{r}{2} + e(d-r)}$$

for $r \in \{0, 1, \dots, d\}$. Here $W_0 = \langle j \rangle$, where j is the all-one vector. Furthermore, notice that all eigenspaces W_i of an association scheme are pairwise orthogonal (see [2, Ch. 2]).

4. An Algebraic Bound

We shall apply a technique that was, to the knowledge of the author, first used by Willem H. Haemers in [8]. The author learned about this technique from a paper by Tokushige [17], where he uses a variant of the result based on the work of Ellis, Friedgut, and Pipel [5] to prove a result on EKR sets of permutations. Let G be a graph with n vertices $\{1, \dots, n\}$. A matrix $A = (a_{xy}) \in \mathbb{C}^{n \times n}$ is called *extended weight adjacency matrix* of G if A is symmetric, and

- (1) $a_{xy} \leq 0$ if x and y are non-adjacent,
- (2) $a_{xx} = 0$,
- (3) the all-ones vector j is an eigenvector of A .
- (4) A is not the all-zero matrix.

Let $\lambda_1, \dots, \lambda_n$ be the (possibly pairwise equal) eigenvalues of A . Denote the eigenvalue of j by k . Denote the smallest eigenvalue by λ_- , and the corresponding eigenspace by V_- . Denote the largest eigenvalue with eigenvectors not in $\langle j \rangle$ by λ_+ . Denote the corresponding eigenspace by V_+ if $k \neq \lambda_{\max}$, and by $\langle j \rangle \perp V_+$ if $k = \lambda_{\max}$. Denote $\max\{-\lambda_-, \lambda_+\}$ by λ_b . We say that λ_b is the *second largest absolute eigenvalue* of A .

A *characteristic vector* χ_Y of a subset $Y \subseteq \{1, \dots, n\}$ is defined by

$$\chi_i = \begin{cases} 1 & \text{if } i \in Y \\ 0 & \text{if } i \notin Y \end{cases}.$$

Ellis, Friedgut, and Pipel used the following result, a generalization of the Hoffman bound for cliques in graphs:

LEMMA 4.1. *Let A be an extended weight adjacency matrix of a regular graph G with n vertices. Let k be the eigenvalue of the all-one vector j , and let λ_b the second largest absolute eigenvalue of A . Let Y and Z be sets of vertices such that there are no edges between Y and Z . Then*

$$\sqrt{|Y| \cdot |Z|} \leq \frac{\lambda_b}{k + \lambda_b} n.$$

REMARK 4.2. Very often the Hoffman bound is only formulated for so-called *weight adjacency matrices* or *pseudo adjacency matrices* where a_{ij} is zero if i and j are non-adjacent and $a_{ij} > 0$ if i and j are adjacent. It is regularly mentioned in the literature and easy to see that all the proofs for variants of the Hoffman bound (at least the ones used in this paper) also work for extended adjacency matrices without changing much of the proof. The Hoffman bound for an extended weight matrix A is optimal only if A is also a weight matrix. Our more general definition will turn out to be more convenient in Section 6 where we shall not bother to calculate the exact minimum of the Hoffman bound.

Tokushige reformulates Lemma 4.1 in a more detailed way in [17], Lemma 2. Unfortunately, his reformulation misses to point out some details necessary for the special case handled in this paper. Hence, we have to restate his wording of the following lemma.

LEMMA 4.3. *Suppose that equality holds in Lemma 4.1. Then one of the following cases occurs:*

- (a) *We have $\lambda_+ = \lambda_b > -\lambda_-$, $\chi_Y = \alpha j + v_+$, and $\chi_Z = \alpha j - v_+$ for some vector $v_+ \in V_+$.*
- (b) *We have $\lambda_+ < \lambda_b = -\lambda_-$, $\chi_Y = \alpha j + v_-$, and $\chi_Z = \alpha j + v_-$ for some vector $v_- \in V_-$. In this case $Y = Z$, and Y is an EKR set.*
- (c) *We have $\lambda_+ = \lambda_b = -\lambda_-$, $\chi_Y = \alpha j + v_- + v_+$, and $\chi_Z = \alpha j + v_- - v_+$ for some vectors $v_- \in V_-$ and $v_+ \in V_+$.*

Furthermore, $|Y| = |Z| = \alpha n$.

PROOF. The proof of Lemma 2 in [17] still works for the first three claims if one reads it carefully. For the claim $|Y| = |Z|$ consider the following: j is orthogonal to v_- and v_+ . Hence,

$$|Y| = \chi_Y^T j = \alpha n = \chi_Z^T j = |Z|.$$

□

5. Cross-intersecting EKR Sets of Maximum Size

In this section we shall calculate tight upper bounds for all polar spaces except $H(2d-1, q^2)$, and classify all examples in case of equality. For all polar spaces except $H(2d-1, q^2)$ we can imitate the approach of Pepe, Storme, and Vanhove [16]. Recall from Section 3 that we have a natural ordering of the eigenspaces $W_0(=\langle j \rangle), W_1, \dots, W_d$ of the association scheme which we defined on generators of a polar space of rank d .

THEOREM 5.1. *Let (Y, Z) be a cross-intersecting EKR set of generators of a polar space \mathcal{P} . Let n be the number of generators of \mathcal{P} . Then we have the following:*

- *If $\mathcal{P} = Q^+(2d-1, q)$, then $\sqrt{|Y| \cdot |Z|}$ is at most $n/2$, and if this bound is reached, then $\chi_Y, \chi_Z \in W_0 \perp W_d$.*
- *If $\mathcal{P} \in \{Q(2d, q), W(2d-1, q)\}$, then $\sqrt{|Y| \cdot |Z|}$ is at most the number of generators on a fixed point, and if this bound is reached, then $\chi_Y, \chi_Z \in W_0 \perp W_1 \perp W_d$.*
- *If $\mathcal{P} \in \{H(2d, q), Q^-(2d+1, q)\}$, then $\sqrt{|Y| \cdot |Z|}$ is at most the number of generators on a fixed point, and if this bound is reached, then $\chi_Y, \chi_Z \in W_0 \perp W_1$.*

PROOF. To apply Lemma 4.1, we have to calculate the second largest absolute eigenvalue of the disjointness graph (with associated adjacency matrix A_d). The eigenvalues of A_d were given in (3.2) as

$$(-1)^r q^{\binom{d-r}{2} + \binom{r}{2} + e(d-r)}.$$

For $r = 0$ this is the eigenvalue which belongs to the all-one vector j , so with k defined as in Lemma 4.1 we have

$$k = q^{\binom{d}{2} + de}.$$

For $e = 0$ note that the absolute eigenvalues for $r = 0$ and $r = d$ are equal. Therefore, the eigenspace belonging to k has dimension at least 2 which make k also the second largest absolute eigenvalue. Hence, we have the following for the different polar spaces. For $e = 0$ (i.e. $\mathcal{P} = Q^+(2d-1, q)$) the second largest absolute eigenvalue occurs if and only if $r = d$, for $e = 1$ (i.e. $\mathcal{P} \in \{Q(2d, q), W(2d-1, q)\}$) the second largest absolute eigenvalue occurs if and only if $r \in \{1, d\}$, for $e \in \{3/2, 2\}$ (i.e. $\mathcal{P} \in \{H(2d, q), Q^-(2d+1, q)\}$) the second largest absolute eigenvalue occurs if and only if $r = 1$. Applying Lemma 4.1 and Lemma 4.3 yields the assertion. □

Using Lemma 4.3 and the classification of EKR sets of generators given in [16] we get the following result.

COROLLARY 5.2. *Let (Y, Z) be an cross-intersecting EKR set of a finite classical polar space \mathcal{P} not isomorphic to $Q(2d, q)$ with d even, $W(2d-1, q)$ with d even, $Q^+(2d-1, q)$ with d even, or $H(2d, q^2)$, where $|Y| \cdot |Z|$ reaches the bound in Theorem 5.1. Then $Y = Z$, and Y is an EKR set.*

PROOF. For the stated cases, the eigenspaces given in Theorem 5.1 (not equal to $\langle j \rangle$) belong to negative eigenvalues. By Lemma 4.3, then all cross-intersecting EKR which reach the bound given in Theorem 5.1 are EKR sets. □

Similar to [16] we shall continue to classify the more complicated cases.

5.1. The Hyperbolic Quadric, d even. The generators of $Q^+(2d-1, q)$ can be partitioned into two sets X_1 and X_2 of generators (commonly known as *latins* and *greeks*) with $|X_1| = |X_2| = n/2$. For $x_1 \in X_1$ and $x_2 \in X_2$ the codimension of the intersection of $x \cap y$ is odd. For $x_1, x_2 \in X_1$ the codimension of the intersection of $x \cap y$ is even. This implies for d even that (X_1, X_2) is a cross-intersecting EKR set of maximum size according to Theorem 5.1. There exist $x_1, x_2 \in X_1$ with $\dim(x_1 \cap x_2) = 0$ if d is even, so (X_1, X_1) is not a cross-intersecting EKR set.

THEOREM 5.3. *Let (Y, Z) be an cross-intersecting EKR set of maximum size of $Q^+(2d-1, q)$ with d even. Then $Y = X_i$ and $Z = X_j$ for $\{i, j\} = \{1, 2\}$.*

PROOF. By Theorem 5.1, we have $\chi_Y, \chi_Z \in W_0 \perp W_d$. As in Theorem 16 of [16] W_0 is spanned by $\chi_{X_1} + \chi_{X_2}$, and W_d is spanned by $\chi_{X_1} - \chi_{X_2}$. Hence $\chi_Y, \chi_Z \in \{\chi_{X_1}, \chi_{X_2}\}$ as $\chi_Y, \chi_Z, \chi_{X_1}, \chi_{X_2}$ are 0-1-vectors with $\chi_{X_1} + \chi_{X_2} = j$. Hence without loss of generality $Y = X_1$. Since (X_1, X_1) is not a cross-intersecting EKR set, we have $Z = X_2$. \square

5.2. The Parabolic Quadric and the Symplectic Polar Space, d even. If a cross-intersecting EKR set (Y, Z) of $Q(2d, q)$ satisfies $\chi_Y, \chi_Z \in W_0 \perp W_1$, then $Y = Z$, and Y is an EKR set as before. So only the case $\chi_Y, \chi_Z \in W_0 \perp W_1 \perp W_d$ remains. In the following denote W_1 by V_- and W_d by V_+ . Furthermore, as in Lemma 4.3 we write

$$\begin{aligned}\chi_Y &= \alpha j + v_- + v_+ \\ &= \frac{|Y|}{n} j + v_- + v_+ \\ &= \frac{\lambda_b}{k + \lambda_b} j + v_- + v_+\end{aligned}$$

and

$$\begin{aligned}\chi_Z &= \alpha j + v_- + v_+ \\ &= \frac{|Z|}{n} j + v_- + v_+ \\ &= \frac{\lambda_b}{k + \lambda_b} j + v_- - v_+\end{aligned}$$

with $v_- \in V_-$ and $v_+ \in V_+$. We need the following well-known lemma.

LEMMA 5.4. *Let $\chi \in \langle j \rangle \perp V$ for some eigenspace V of an (extended weight) adjacency matrix of a k -regular graph with n vertices associated with eigenvalue λ . Then the characteristic vector e_i of the i -th vertex satisfies*

$$e_i^T A \chi = \frac{\chi^T j}{n} (k - \lambda) + \lambda e_i^T \chi.$$

PROOF. As $\chi \in \langle j \rangle \perp V$, we can write $\chi = \alpha j + v$ for some $v \in V$ and $\alpha = \frac{\chi^T j}{n}$. Then

$$\begin{aligned}e_i^T A \chi &= e_i^T A(\alpha j + v) \\ &= e_i^T (\alpha k j + \lambda v) \\ &= e_i^T (\alpha(k - \lambda)j + \lambda \chi) \\ &= \frac{\chi^T j}{n} (k - \lambda) + \lambda e_i^T \chi.\end{aligned}$$

\square

COROLLARY 5.5. *Let $\chi, \psi \in \langle j \rangle \perp V_- \perp V_+$ for some eigenspaces V_- , respectively, V_+ of a (extended weight) adjacency matrix of the graph with eigenvalue λ_- , respectively, λ_+ . If $\chi = \alpha j + v_- + v_+$ and $\psi = \alpha j + v_- - v_+$ for some $\alpha \in \mathbb{R}$, $v^- \in V_-$, and $v^+ \in V^+$, then*

$$\begin{aligned} e_i^T A \chi &= \frac{(\chi + \psi)^T j}{2n} (k - \lambda_-) + \frac{(\lambda_- + \lambda_+)}{2} e_i^T \chi \\ e_i^T A \psi &= \frac{(\chi + \psi)^T j}{2n} (k - \lambda_-) + \frac{(\lambda_- - \lambda_+)}{2} e_i^T \psi \end{aligned}$$

PROOF. We have $\chi + \psi \in \langle j \rangle \perp V_-$ and $\chi - \psi \in V_+$. By Lemma 5.4 and $j^T v^- = 0 = j^T v^+$,

$$\begin{aligned} e_i^T A(\chi + \psi) &= \frac{(\chi + \psi)^T j}{n} (k - \lambda_-) + \lambda_- e_i^T (\chi + \psi) \\ e_i^T A(\chi - \psi) &= \lambda_+ e_i^T (\chi - \psi). \end{aligned}$$

Now the equations $2e_i^T A \chi = e_i^T A(\chi + \psi) + e_i^T A(\chi - \psi)$ and $2e_i^T A \psi = e_i^T A(\chi + \psi) - e_i^T A(\chi - \psi)$ yield the assertion. \square

LEMMA 5.6. *For the adjacency matrix A_{d-s} , $0 < s < d$, the eigenspace W_1 is associated with eigenvalue*

$$\lambda_{-,s} := -\begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}},$$

the eigenspace W_d is associated with eigenvalue

$$\lambda_{+,s} := (-1)^{d-s} \begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}},$$

and

$$k_s := \begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s+1}{2}} = \left(\begin{bmatrix} d-1 \\ s \end{bmatrix} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{d-s} \right) q^{\binom{d-s+1}{2}}.$$

PROOF. See [18, Theorem 4.3.6], where the eigenvalue of W_j for A_i is given by

$$\sum_{0, j-i \leq u \leq d-i, j} (-1)^{j+u} \begin{bmatrix} d-j \\ d-i-u \end{bmatrix} \begin{bmatrix} j \\ u \end{bmatrix} q^{(u+i-j)(u+i-j+2e-1)/2 + \binom{j-u}{2}}.$$

For $j = 1$, respectively, $j = d$, and $i = d - s$, this formula yields the assertion. The last equality is an application of (2.1). \square

PROPOSITION 5.7. *Let (Y, Z) be a cross-intersecting EKR set of $Q(2d, q)$ or $W(2d-1, q)$, d even, of maximum size such that $Y \cap Z \neq \emptyset$. Let $G \in Y$.*

- (a) *If $d - s$ is even, then G meets 0 elements of Z in dimension s .*
- (b) *If $d - s$ is odd, then G meets 0 elements of Y in dimension s .*
- (c) *If $d - s$ is even, then G meets*

$$\begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}}$$

elements of Y in dimension s .

- (d) *If $d - s$ is odd, then G meets*

$$\begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}}$$

elements of Z in dimension s .

In particular, $Y \cap Z = \emptyset$.

PROOF. We can calculate these numbers with Lemma 5.6 and Corollary 5.5 by choosing $\chi_{\{G\}}$ as e_i . For A_{d-s} the parameters are given by

$$\begin{aligned} k_s - \lambda_{-,s} &= \left(\begin{bmatrix} d-1 \\ s \end{bmatrix} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{d-s} \right) q^{\binom{d-s+1}{2}} \\ &\quad + \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}} - \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}} \\ &= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^{d-s} + 1) + q^{\binom{d-s+1}{2}} \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} (q^{d-s} - 1) \\ &\stackrel{\text{Def.}}{=} q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^{d-s} + 1) + q^{d-s} \cdot q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^s - 1) \\ &= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^d + 1), \end{aligned}$$

for $d-s$ even

$$\begin{aligned} \lambda_{-,s} + \lambda_{+,s} &\stackrel{(2.1)}{=} 2 \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}}, \\ \lambda_{-,s} - \lambda_{+,s} &\stackrel{(2.1)}{=} -2 \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}}, \end{aligned}$$

for $d-s$ odd

$$\begin{aligned} \lambda_{-,s} + \lambda_{+,s} &\stackrel{(2.1)}{=} -2 \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}}, \\ \lambda_{-,s} - \lambda_{+,s} &\stackrel{(2.1)}{=} 2 \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}}. \end{aligned}$$

By assumption (Y, Z) is of maximum size, so by Lemma 4.3 (recall $\lambda_+ = \lambda_{+,0}$, and $\lambda_- = \lambda_{-,0}$)

$$\chi_Z^T j = |Z| = |Y| = \chi_Y^T j = \frac{n\lambda_+}{k + \lambda_+}.$$

Hence,

$$\frac{(\chi_Y + \chi_Z)^T j}{2n} = \frac{\lambda_+}{k + \lambda_+} = \frac{q^{\binom{d}{2}}}{q^{\binom{d+1}{2}} + q^{\binom{d}{2}}} = \frac{1}{q^d + 1}.$$

Hence by Corollary 5.5,

$$\begin{aligned} e_i^T A\chi &= \frac{k_s - \lambda_{-,s}}{q^d + 1} + \frac{(\lambda_- + \lambda_+)}{2} e_i^T \chi \\ &= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} + \frac{(\lambda_- + \lambda_+)}{2} e_i^T \chi. \end{aligned}$$

If $d-s$ is even and $e_i^T \chi = 1$, then by (2.1)

$$e_i^T A\chi = q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} + q^{\binom{d-s+1}{2}} \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} = q^{\binom{d-s}{2}} \begin{bmatrix} d \\ s \end{bmatrix}.$$

If $d-s$ is odd and $e_i^T \chi = 1$, then

$$e_i^T A\chi = q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} - q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} = 0.$$

All the remaining cases are either similar or trivial. \square

Now we have a strong combinatorial information about cross-intersecting EKR sets (Y, Z) of maximum size which are not EKR sets. By adding some geometrical arguments this leads to a complete classification of cross-intersecting EKR sets in these parabolic and symplectic polar spaces as we shall see in the following.

LEMMA 5.8. *Let (Y, Z) be a cross-intersecting EKR set of $Q(2d, q)$ or $W(2d - 1, q)$, d even, of maximum size. Let $G, H \in Y$ disjoint (see Proposition 5.7 (c)). Let $\pi_1, \dots, \pi_{[d]} \subseteq G$ be the $[d]$ subspaces of dimension $d - 1$ of G . Then the following holds:*

- (a) *Exactly $[d]$ elements $z_1, \dots, z_{[d]}$ of Z meet G in dimension $d - 1$.*
- (b) *We have $\{z_i \mid i \in \{1, \dots, [d]\}\} = \{\langle \pi_i, \pi_i^\perp \cap H \rangle \mid i \in \{1, \dots, [d]\}\}$.*

PROOF. By Proposition 5.7 (d) and $Y \neq Z$, exactly $[d]$ elements of Z meet Y in dimension $d - 1$. This shows (a). By Proposition 5.7 (b), $\dim(z_i \cap z_j) < d - 1$ for $i \neq j$. Hence, each hyperplane π_i lies in exactly one element z_j , and z_j satisfies $z_j \subseteq \pi_i^\perp$. Since (Y, Z) is a cross-intersecting EKR set, all z_j meet H in at least a point. Since $\pi_i \subseteq G$ and $G \cap H = \emptyset$, we see that $\pi_i^\perp \cap H$ is a point. Hence,

$$\{z_i \mid i \in \{1, \dots, [d]\}\} = \{\langle \pi_i, \pi_i^\perp \cap H \rangle \mid i \in \{1, \dots, [d]\}\}.$$

□

We write

$$(5.9) \quad \begin{aligned} Z_{G,H} &= \{\{z_i \mid i \in \{1, \dots, [d]\}\} \\ &= \{\langle \pi_i, \pi_i^\perp \cap H \rangle \mid i \in \{1, \dots, [d]\}\} \subseteq Z \end{aligned}$$

for $G, H \in Y$ whenever Lemma (5.8) is applicable.

5.2.1. *The Parabolic Quadric $Q(2d, q)$.* Let h be a subspace of $PG(2d, q)$. We write $Y \subseteq h$ if all elements of Y are subspaces of h , $Y \cap h$ for all elements of Y in h , and $Y \setminus h$ for all elements of Y not in h .

LEMMA 5.10. *Let G and H be disjoint generators of $Q(2d, q)$. Then $h := \langle G, H \rangle \cap Q(2d, q)$ is isomorphic to $Q^+(2d - 1, q)$.*

PROOF. The generators G and H are disjoint, hence $h \cap Q(2d, q)$ is not degenerate. The hyperplane h obviously contains generators, hence $h \cap Q(2d, q)$ does not have type $Q^-(2(d - 1) + 1, q)$. Therefore, the intersection $h \cap Q(2d, q)$ is isomorphic to $Q^+(2d - 1, q)$. □

LEMMA 5.11. *Let (Y, Z) be a cross-intersecting EKR set of $Q(2d, q)$, d even, of maximum size such that $Y \cap Z \neq Y$. Let $G \in Y$. Let \tilde{Y} be the set of the $q^{\binom{d}{2}}$ generators of Y disjoint to G (see Proposition 5.7).*

- (a) *There exists a hyperplane h of type $Q^+(2d - 1, q)$ such that $G, \tilde{Y} \subseteq h$.*
- (b) *If $\tilde{G} \in Y \setminus h$, then \tilde{G} meets all elements of \tilde{Y} non-trivially.*
- (c) *If $\tilde{G} \in Y$ and $\dim(\tilde{G} \cap H) = d - 2$ for any $H \in \tilde{Y}$, then \tilde{G} and the $q^{\binom{d}{2}}$ generators disjoint to \tilde{G} are in h .*

PROOF. By Proposition 5.7, a generator $G \in Y$ is disjoint to $q^{\binom{d}{2}}$ generators of Y . Let $H \in \tilde{Y}$. By Lemma 5.10, $h := \langle G, H \rangle$ has type $Q^+(2d - 1, q)$. We shall show $\tilde{Y} \subseteq h$.

Suppose to the contrary that there exists a generator $\tilde{H} \in \tilde{Y}$ not in h . We define Z_G in (5.9) by

$$\begin{aligned} Z_G &= Z_{G,H} \\ &= \{z_i \mid i \in \{1, \dots, [d]\}\} \subseteq Z. \end{aligned}$$

Set

$$\mathcal{P} := \{\tilde{H} \cap z_i \mid i \in \{1, \dots, [d]\}\}.$$

The generators of Z_G intersect G in a hyperplane of G , hence \mathcal{P} is a set of points. For $i \neq j$ we have $G \subseteq \langle z_i \cap G, z_j \cap G \rangle^\perp$, so $(z_i \cap z_j) \setminus G$ is empty. Hence, $|\mathcal{P}| = |Z_G| = [d]$. Furthermore, $Z_G \subseteq h$, so $\mathcal{P} \subseteq \tilde{H} \cap h$. Hence,

$$[d] = |\mathcal{P}| \leq |\tilde{H} \cap h| = [d-1].$$

This is a contradiction. Thus, $\tilde{Y} \subseteq h$. This proves (a).

Assume that there exists a generator $\tilde{G} \in Y \setminus h$ with $\dim(H \cap \tilde{G}) = 0$. Then, by (a), H is only disjoint to generators of Y in $\langle \tilde{G}, H \rangle$. Hence, $\langle \tilde{G}, H \rangle = \langle G, H \rangle$. This contradicts $\tilde{G} \in Y \setminus h$. This proves (b).

Suppose that there exists a generator $\tilde{G} \in Y \setminus h$ with $\dim(H \cap \tilde{G}) = d-2$. By (b) and Proposition 5.7, $\dim(G \cap \tilde{G}) \geq 2$. Hence,

$$\tilde{G} = \langle H \cap \tilde{G}, G \cap \tilde{G} \rangle \subseteq h,$$

which contradicts $\tilde{G} \notin h$. This shows $\tilde{G} \in \tilde{Y}$. By (a), all elements disjoint to \tilde{G} are in $\langle G, \tilde{G} \rangle = h$. \square

We need the following bound.

LEMMA 5.12. *Let $q \geq 2$. Let $d \geq 1$. Then*

$$\prod_{i=1}^{d-1} (q^i + 1) \leq \frac{2q^d}{q^d + 1} \left(q^{\binom{d}{2}} - q^{\binom{d-1}{2}} + 1 \right) + q^{\binom{d-2}{2} + 2(d-2)}.$$

PROOF. We will prove the assertion by induction over d . It can be easily checked that the assertion is true for $d \leq 4$. If the assertion is true for $d \geq 4$, then

$$\begin{aligned} \prod_{i=1}^d (q^i + 1) &\leq (q^d + 1) \left(\frac{2q^d}{q^d + 1} \left(q^{\binom{d}{2}} - q^{\binom{d-1}{2}} + 1 \right) + q^{\binom{d-2}{2} + 2(d-2)} \right) \\ &\stackrel{(*)}{\leq} \frac{2q^{d+1}}{q^{d+1} + 1} \left(q^{\binom{d+1}{2}} - q^{\binom{d}{2}} + 1 \right) + q^{\binom{d-1}{2} + 2(d-1)}. \end{aligned}$$

The difference between the right hand side of (*) and the left hand side of (*) equals

$$\begin{aligned} &\frac{q^{-\frac{3d}{2}-1}}{q^d + 1} (2q^{\frac{d^2}{2} + 2d + 3} - 2q^{\frac{d^2}{2} + 2d + 2} - 3q^{\frac{d^2}{2} + 2d + 1} \\ &\quad + 2q^{\frac{d^2}{2} + d + 2} - q^{\frac{d^2}{2} + d} - 2q^{\frac{7d}{2} + 2} + 2q^{\frac{5d}{2} + 2} - 2q^{\frac{5d}{2} + 1}) \end{aligned}$$

which is a positive expression for $q \geq 2$ and $d \geq 4$. \square

PROPOSITION 5.13. *Let (Y, Z) be a cross-intersecting EKR set of $Q(2d, q)$ of maximum size such that $Y \cap Z \neq Y$. Then there exists a hyperplane h such that $Y, Z \subseteq h$.*

PROOF. In the view of Lemma 5.11, we find a hyperplane h that contains G , the $q^{\binom{d}{2}}$ generators of Y disjoint to G , and, by Proposition 5.7 and Lemma 5.11 (d), the $\begin{bmatrix} d \\ 2 \end{bmatrix} q^{\binom{d-2}{2}}$ generators which meet G in dimension $d-2$.

Suppose that there exists an element $G \in Y$ that is not in h . Then G and the $q^{\binom{d}{2}}$ elements of Y disjoint to G lie in a second hyperplane $h' \neq h$ by Lemma 5.11. Lemma 5.11 makes it clear that the at least $q^{\binom{d}{2}} + 1$ generators of Y in h are different to the at least $q^{\binom{d}{2}} + \begin{bmatrix} d \\ 2 \end{bmatrix} q^{\binom{d-2}{2}} + 1$ generators of Y in h' .

Hence,

$$|Y| \geq 2 \left(q^{\binom{d}{2}} + 1 \right) + \begin{bmatrix} d \\ 2 \end{bmatrix} q^{\binom{d-2}{2}}.$$

According to Theorem 5.1,

$$|Y| = \prod_{i=1}^{d-1} (q^i + 1).$$

This contradicts Lemma 5.12. \square

THEOREM 5.14. *Let (Y, Z) be a cross-intersecting EKR set of $Q(2d, q)$, or $W(2d - 1, q)$, q even, of maximum size such that $Y \cap Z \neq Y$. Then either $Y = Z$ and Y is an EKR set, or d even and $Y \cup Z$ are the generators of a subgeometry isomorphic to $Q^+(2d - 1, q)$.*

PROOF. First consider $Q(2d, q)$. By Proposition 5.13, $Y, Z \subseteq h$ for some hyperplane h isomorphic to $Q^+(2d - 1, q)$ if not $Y = Z$. Hence, (Y, Z) is a cross-intersecting set of $Q^+(2d - 1, q)$ of maximum size. These sets were classified in Theorem 5.3.

The part of the assertion for $W(2d - 1, q)$, q even, follows, since then $Q(2d, q)$ and $W(2d - 1, q)$ are isomorphic for q even. \square

5.2.2. The Symplectic Polar Space $W(2d - 1, q)$, d even, q odd. Similar to [16] we use the following property of $W(2d - 1, q)$, d even ([16, Theorem 34]):

THEOREM 5.15. *Let ℓ_1, ℓ_2, ℓ_3 be three pairwise disjoint lines of $W(3, q)$, q odd. Then the number of lines meeting ℓ_1, ℓ_2, ℓ_3 is 0 or 2.*

THEOREM 5.16. *Let (Y, Z) be a cross-intersecting EKR set of maximum size of $W(2d - 1, q)$, d even, q odd. Then $Y = Z$.*

PROOF. Suppose to the contrary that $Y \cap Z \neq Y$. By Proposition 5.7, we can find two disjoint generators G and H in Y . Again by Proposition 5.7, there are exactly $q[d][d - 1]/(q + 1)$ generators $Y' \subseteq Y$ which meet G in a subspace of dimension $d - 2$. The generator G has $[d][d - 1]/(q + 1)$ subspaces of dimension $d - 2$. Hence, we find a subspace $\ell \subseteq G$ of dimension $d - 2$ such that ℓ is contained in q elements of Y' . Since q is odd, there are at least three elements y_1, y_2, y_3 of Y through ℓ .

Consider the quotient geometry W_3 of ℓ isomorphic to $W(3, q)$ and the projection of the elements of Y and Z onto W_3 from ℓ . Since elements of Y do not meet each other in dimension $d - 1$ by Proposition 5.7, y_1, y_2, y_3 are three disjoint lines in W_3 after projection. Lemma 5.8 says that the $[d]$ generators $Z' \subseteq Z$ which meet H in dimension $d - 1$ also meet G in $[d]$ pairwise different points. Hence there are at least 3 generators in Z' which are projected onto three different lines on W_3 . These three lines have to meet the projections of y_1, y_2 , and y_3 , since (Y, Z) is an cross-intersecting EKR set. By Theorem 5.15 this is not possible. Contradiction. \square

6. The Hermitian Polar Space $H(2d - 1, q^2)$

It is well-known (see for example [14]) that the linear programming bound given in [13] can be reformulated as a weighted Hoffman bound. Hence, Lemma 4.1 is applicable if $d > 1$. The original bound on EKR sets on $H(2d - 1, q^2)$ is as follows.

THEOREM 6.1 ([13]). *Let Y be a EKR set of $H(2d - 1, q^2)$ with $d > 1$ odd. Then*

$$|Y| \leq \frac{nq^{d-1} - f_1(q^{d-1} - 1)(1 - c)}{q^{2d-1} + q^{d-1} + f_1(q^{d-1} - 1)c} \approx q^{d^2-2d+2},$$

where $n = \prod_{i=0}^{d-1} (q^{2i+1} + 1)$, $f_1 = q^2[d]_{q^2} \frac{q^{2d-3}+1}{q+1}$ and $c = \frac{q^2-q-1+q^{-2d+3}}{q^{2d-1}}$.

The result by Luz [14] which shows that the linear programming bound is a special case of the weighted Hoffman bound¹ also holds for cross-intersecting EKR sets, but we feel that we should show this directly, since the transition from the linear programming technique used in [13] to the weighted Hoffman bound is not obvious. We shall prove a cross-intersecting result similar to [13] in the following.

THEOREM 6.2. *Let (Y, Z) be a cross-intersecting EKR set of $H(2d-1, q^2)$ with $d > 1$. Then*

$$\sqrt{|Y| \cdot |Z|} \leq \frac{n\lambda_b}{\lambda_b - k} \approx q^{d^2-2d+2},$$

where $n = \prod_{i=0}^{d-1} (q^{2i+1} + 1)$, $\lambda_b = -q^{(d-1)^2} - \alpha(1 - f_1 \frac{1-c}{n})$, $k = q^{d^2} + \alpha f_1(c + \frac{1-c}{n})$, $f_1 = q^2[d]_{q^2} \frac{q^{2d-3}+1}{q+1}$, $c = \frac{q^2-q-1+q^{-2d+3}}{q^{2d}-1}$, and

$$\alpha = \begin{cases} q^{d(d-1)} + q^{(d-1)^2} & \text{if } d \text{ odd,} \\ \frac{nq^{d^2-d} - nq^{(d-1)^2}}{n+(2c-2)f_1} & \text{if } d \text{ even.} \end{cases}$$

PROOF. Let $d > 1$. Let A_d be the disjointness matrix as defined in Section 3. Consider the matrix A defined as

$$A = A_d - \alpha E_1 + \frac{\alpha f_1 c}{n} J + \alpha f_1 \frac{1-c}{n} I.$$

Claim 1. Our first claim is that A is a extended weight adjacency matrix. By Section 3, it is clear that the entry (x, y) of E_1 equals $Q_{i,1}/n$ if x and y meet in codimension i . It was shown in [13, Equations (6)–(11)] that the following holds (note that the equation in [13] do not depend on d odd):

- (a) $Q_{0,1} = f_1$,
- (b) $Q_{d-1,1} = f_1 c$,
- (c) $Q_{s,1} \geq f_1 c$ if $s < d$,
- (d) $Q_{d,1} < 0$.

Hence, the entry (x, y) of the matrix A is 0 if $x = y$, it is less or equal to zero if $1 \leq \text{codim}(x \cap y) \leq d-1$, and it is larger than 1 if x and y are disjoint. This shows that A is an extended weight adjacency matrix of the disjointness graph of generators.

Claim 2. Our second claim is that one of the second absolute largest eigenvalues of A is

$$-q^{(d-1)^2} - \alpha \left(1 - f_1 \frac{1-c}{n} \right),$$

and that

$$k = q^{d^2} + \alpha f_1 \left(c + \frac{1-c}{n} \right).$$

By (3.2), the eigenvalues of A are

$$\begin{aligned} & q^{d^2} + \alpha f_1 \left(c + \frac{1-c}{n} \right) \text{ for } \langle j \rangle, \\ & -q^{(d-1)^2} - \alpha \left(1 - f_1 \frac{1-c}{n} \right) \text{ for } W_1, \\ & (-1)^r q^{(d-r)^2+r(r-1)} + \alpha f_1 \frac{1-c}{n} \text{ for } W_r \text{ with } 1 < r < d, \\ & (-1)^d q^{d(d-1)} + \alpha f_1 \frac{1-c}{n} \text{ for } W_d. \end{aligned}$$

¹This seems to be part of the mathematical folklore for a long time, but the author is not aware of any source older than [14].

An simple calculation shows that $-q^{(d-1)^2} - \alpha(1 - f_1 \frac{1-c}{n}) = (-1)^d(q^{d(d-1)} - \alpha f_1 \frac{1-c}{n})$ is the second largest absolute eigenvalue. This proves our claim.

Now we can apply Lemma 4.1 with these values. Note that k has approximately size q^{d^2+d-2} , the second largest absolute eigenvalue λ_b has approximately size $q^{d(d-1)}$, and n has approximately size q^{d^2} . Therefore,

$$\frac{n\lambda_b}{\lambda_b - k}$$

has approximately size q^{d^2-2d+2} . \square

Note that the normal adjacency matrix of the graph only yields q^{d^2-d} as an upper bound, so this improves the bound significantly.

For the sake of completeness we want to mention the cross-intersecting EKR sets for $d = 2$ is We will do this after providing a general geometrical results on (maximal) cross-intersecting EKR sets, where we call an (cross-intersecting) EKR set (Y, Z) *maximal* if there exists no generator x such that $(Y \cup \{x\}, Z)$ or $(Y, Z \cup \{x\})$ is an cross-intersecting EKR set.

LEMMA 6.3. *Let (Y, Z) be a maximal cross-intersecting EKR set in a finite classical polar space of rank d . If two distinct elements $y_1, y_2 \in Y$ meet in a subspace of dimension $d - 1$, then all elements of Z meet this subspace in at least a point.*

PROOF. Assume that there exists a generator z which meets y_1 and y_2 in points P, Q not in $y_1 \cap y_2$. Then $\langle P, Q, y_1 \cap y_2 \rangle$ is a totally isotropic subspace of dimension $d + 1$. Contradiction. \square

Lemma 4.1 yields

$$\frac{(q+1)(q^3+1)}{q^2+1}$$

as an upper bound for $H(3, q^2)$. This bound is not sharp as the following trivial results shows.

THEOREM 6.4. *Let (Y, Z) be a maximal cross-intersecting EKR set of $H(3, q^2)$ with $|Y| \geq |Z|$. Then one of the following cases occurs:*

- (a) *The set Y is the set of all lines of $H(3, q^2)$, and $Z = \emptyset$. Here $|Y| \cdot |Z| = 0$.*
- (b) *The set Y is the set of all lines meeting a fixed line ℓ in at least a point, and $Z = \{\ell\}$. Here $|Y| \cdot |Z| = (q^2 + 1)q + 1$.*
- (c) *The set Y is the set of all lines on a fixed point P , and $Y = Z$. Here $|Y| \cdot |Z| = (q + 1)^2$.*
- (d) *The set Y is the set of lines meeting two disjoint lines ℓ_1, ℓ_2 , and $Z = \{\ell_1, \ell_2\}$. Here $|Y| \cdot |Z| = 2(q^2 + 1)$.*
- (e) *The set Y is the set of lines meeting three disjoint lines ℓ_1, ℓ_2, ℓ_3 , and Z is the set of all $q + 1$ lines meeting the lines of Y . Here $|Y| \cdot |Z| = (q + 1)^2$.*

PROOF. Assume that (a) does not occur.

By Lemma 6.3, as soon as two elements of Y meet in a point P , then all elements of Z contain P . Hence, (b) occurs or at least 2 elements of Z meet in P . Hence, all elements of Y contain P by Lemma 6.3. This is case (c).

So assume that Y and vice-versa Z only consist of disjoint lines. If there are two lines $\ell_1, \ell_2 \subseteq Z$, then there are $q^2 + 1$ (disjoint) lines L meeting ℓ_1 and ℓ_2 in a point (hence $|Y| \leq q^2 + 1$). If more than $q + 1$ of these lines meet ℓ_1 (hence $|Y| > q + 1$), then Z contains at most two lines, since in $H(3, q^2)$ exactly $q + 1$ lines meet 3 pairwise disjoint lines in a point. This yields (d). If $|Z| \geq 3$, then $|Y| \leq q + 1$ by the previous argument. We may assume $|Y| \geq 3$. Then it is well-known that there are exactly $q + 1$ lines meeting the $q + 1$ lines of Y . Hence, we can add these lines and then Z is maximal. This yields (e). \square

The author tried to prove that the maximum cross-intersecting EKR set of $H(5, q^2)$ is the unique EKR of maximum size given in [16], but aborted this attempt after he got lost in too many case distinctions. This EKR set of all generators meeting a fixed generator in at least a line is the largest cross-intersecting EKR set known to the author and has size $q^5 + q^3 + q + 1$. The largest example known to the author for $H(7, q^2)$ is the following.

EXAMPLE 6.5. Let G be a generator of $H(7, q^2)$. Let Y be the set of all generators that meet G in at least a 2-space. Let Z be the set of all generators that meet G in at least a 3-space. Then (Y, Z) is a cross-intersecting EKR set.

PROOF. A generator of $H(7, q^2)$ is a 4-space. A plane and a line of a 4-space meet pairwise in at least a point. Hence, (Y, Z) is a cross-intersecting EKR set. \square

In this example Y has

$$1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + q^{10} + q^{12}$$

elements, Z has

$$1 + q + q^3 + q^5 + q^7$$

elements, so in total the cross-intersecting EKR set has size

$$\sqrt{|Y| \cdot |Z|} \approx q^{19/2}.$$

The bound given in Theorem 6.2 for this case is approximately q^{10} . For $H(2d-1, q^2)$, $d > 4$, the largest example known to the author is the EKR set of all generators on a fixed point. The author assumes that the largest known examples are also the largest examples.

7. Summary

We summarize our results in the following table. We only list the cases, where cross-intersecting EKR sets of maximum size are not necessarily EKR sets. The table includes the size of the largest known example if it is not known if the best known bound does not seem to be tight.

Polar Space	Maximum Size $\sqrt{ Y \cdot Z }$	Largest (known) Ex- amples	Reference
$Q^+(2d-1, q)$, d odd	$n/2$	Y latins, Z greeks	Th. 5.3
$Q(2d, q)$, d odd	$(q+1) \cdot \dots \cdot (q^{d-1} + 1)$	Y latins and Z greeks of a $Q^+(2d+1, q)$, or $Y = Z$ EKR set	Th. 5.14
$W(2d-1, q)$, d odd, q even	$(q+1) \cdot \dots \cdot (q^{d-1} + 1)$	see $Q(2d, q)$	Th. 5.14
$H(3, q^2)$	$q^3 + q + 1$	Th. 6.4	Th. 6.4
$H(5, q^2)$	$\lesssim q^5$	largest EKR set, size $\approx q^5$	Th. 6.2
$H(7, q^2)$	$\lesssim q^{10}$	Example 6.5, size $\approx q^{19/2}$	Th. 6.2
$H(2d-1, q^2)$, $d > 1$	$\lesssim q^{(d-1)^2+1}$	all generators on a point, size $\approx q^{(d-1)^2}$	Th. 6.2

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